INVERSE TRIGONOMETRIC FUNCTIONS

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If $\sin\theta = x$ is a trigonometrical equation, then the value of θ which satisfies this equation is denoted by $\sin^{-1} x$ and it is read as 'sine inverse x'. It is called inverse function of sine. Similarly inverse functions of other trigonometrical functions are defined. Hence inverse functions of trigonometrical functions are defined as follows-

$\sin^{-1} x = \theta \Leftrightarrow \sin\theta = x$

$\cos^{-1} x = \theta \Leftrightarrow \cos\theta = x$

$\tan^{-1} x = \theta \Leftrightarrow \tan\theta = x$

$\cot^{-1} x = \theta \Leftrightarrow \cot\theta = x$

$\sec^{-1} x = \theta \Leftrightarrow \sec\theta = x$

$\text{cosec}^{-1} x = \theta \Leftrightarrow \text{cosec}\,\theta = x$

DOMAIN AND RANGE OF INVERSE FUNCTIONS

As we know that in direct trigonometric functions, we are given the angle and we calculate the trigonometric ratio (sine, cosine, etc.) or the value at that angle. Also to many values of the angle the value of trigonometric ratio is same e.g., $\tan\theta = 1$ for $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, etc. Inverse trigonometry deals with obtaining the angles, given the value of a trigonometric ratio. In inverse trigonometry some restrictions have been imposed on the angles, and these are based on the principle values of the angles.

The inverse of sine function is defined as $\sin^{-1}x = \theta$ or arc sin $x = \theta$, where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ e.g., $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ and nothing else, although $\sin\frac{5\pi}{6}, \sin\frac{13\pi}{6}$ etc. are also equal to $\frac{1}{2}$, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ only. Note that $\sin^{-1} x \ne \frac{1}{\sin x}\left(\because \frac{1}{\sin x} = (\sin x)^{-1}\right)$

We list below the difinitions of all inverse trigonometric functions with their respective domains and ranges.

Function $y = f(x)$	Domain (permitted value of x)	Range (permitted value of y)
(i) $y = \sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii) $y = \cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
(iii) $y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv) $y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi))$
(v) $y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
(vi) $y = \text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

e.g., $\cos^{-1}\left(-\frac{1}{2}\right)=\frac{2\pi}{3}$; $\tan^{-1}(-1) = -\frac{\pi}{4}$; $\text{cosec}^{-1}(2) = \frac{\pi}{6}$; $\sec^{-1}(-1) = \pi$; $\cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$, etc.

PROPERTIES OF INVERSE FUNCTIONS

(i) $\sin(\sin^{-1}x) = x$ $\quad -1 \le x \le 1$

$\cos(\cos^{-1}x) = x$ $\quad -1 \le x \le 1$

$\tan(\tan^{-1}x) = x$ $\quad -\infty < x < \infty$

$\cot(\cot^{-1}x) = x$ $\quad -\infty < x < \infty$

$\sec(\sec^{-1}x) = x$ $\quad x \le -1$ or $x \ge 1$

$\text{cosec}(\text{cosec}^{-1}x) = x$ $\quad x \le -1$ or $x \ge 1$

(ii) $\sin^{-1}(\sin\theta) = \theta$ only if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\cos^{-1}(\cos\theta) = \theta$ only if $0 \le \theta \le \pi$

$\tan^{-1}(\tan\theta) = \theta$ only if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\cot^{-1}(\cot\theta) = \theta$ only if $0 < \theta < \pi$

$\sec^{-1}(\sec\theta) = \theta$ only if $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

$\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$ only if $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$

e.g., $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) \ne \frac{5\pi}{6}$

$\tan^{-1}\left(\tan\frac{5\pi}{6}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

(iii) $\sin^{-1}(-x) = -\sin^{-1}x$ $\quad -1 \le x \le 1$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$ $\quad -1 \le x \le 1$

$\tan^{-1}(-x) = -\tan^{-1}x$ $\quad -\infty < x < \infty$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$ $\quad -\infty < x < \infty$

(iv) $\sin^{-1}x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$ $\quad -1 \le x < 0$ or $0 < x \le 1$

$\cos^{-1}x = \sec^{-1}\left(\frac{1}{x}\right)$ $\quad -1 \le x < 0$ or $0 < x \le 1$

$\tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)$ only if $x > 0$

because range of these two functions are different.

If $x < 0$, $\tan^{-1}x = -\pi + \cot^{-1}\left(\frac{1}{x}\right)$

(v) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ $\quad -1 \le x \le 1$

$\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$ $\qquad -\infty < x < \infty$

$\sec^{-1} x + \text{cosec}^{-1} x = \dfrac{\pi}{2}$ $\qquad x \le -1$ or $x \ge 1$

SUM AND DIFFERENCE FORMULAE

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2})$ $\qquad$ if $xy < 0$ or $x^2 + y^2 \le 1$

$$= \begin{cases} \pi - \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), & \text{if } x > 0, y > 0, x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), & \text{if } x < 0, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

$\sin^{-1} x - \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} - y\sqrt{1-x^2})$ $\qquad$ if $xy \ge 0$ and $x^2 + y^2 \le 1$

(ii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1} (xy - \sqrt{1-x^2}\,\sqrt{1-y^2})$, $\qquad$ if x + y ³ 0

$= 2\pi - \cos^{-1} (xy - \sqrt{1-x^2}\,\sqrt{1-y^2})$, if $x + y < 0$

(iii) $\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\,\sqrt{1-y^2}\right), & \text{if } x \ge 0, y \ge 0, x \le y \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\,\sqrt{1-y^2}\right), & \text{if } x \ge 0, y \ge 0\ x > y \end{cases}$

(iv) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ if $x, y > 0$ and $xy < 1$

$= \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ if $x, y > 0$ and $xy > 1$

$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$ if $x, y > 0$

(v) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$

SOME IMPORTANT RESULTS

(vi) $2\sin^{-1} x = \sin^{-1} 2x\sqrt{1-x^2}$

(vii) $2\cos^{-1} x = \cos^{-1} (2x^2 - 1)$

(viii) $2\tan^{-1} x = \tan^{-1} \dfrac{2x}{1-x^2} = \sin^{-1}\dfrac{2x}{1+x^2} = \cos^{-1}\dfrac{1-x^2}{1+x^2}$

(ix) $3\sin^{-1} x = \sin^{-1} (3x - 4x^3)$

(x) $3\cos^{-1} x = \cos^{-1} (4x^3 - 3x)$

(xi) $3\tan^{-1} x = \tan^{-1} \dfrac{3x - x^3}{1 - 3x^2}$

MISCELLENEOUS RESULTS

(i) $\tan^{-1}\left[\dfrac{x}{\sqrt{a^2-x^2}}\right]=\sin^{-1}\left(\dfrac{x}{a}\right)$

(ii) $\tan^{-1}\left[\dfrac{3a^2x-x^3}{a(a^2-3x^2)}\right]=3\tan^{-1}\left(\dfrac{x}{a}\right)$

(iii) $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]=\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2$

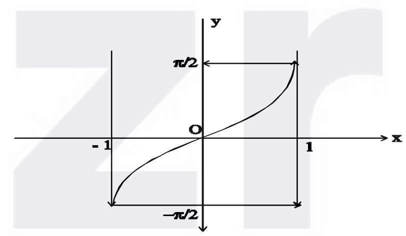
(iv) $\sin^{-1}(x)=\cos^{-1}\left(\sqrt{1-x^2}\right)=\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)=\cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)=\sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)=\operatorname{cosec}^{-1}\left(\dfrac{1}{x}\right)$

(v) $\cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right)=\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)=\cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)=\sec^{-1}\left(\dfrac{1}{x}\right)=\operatorname{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

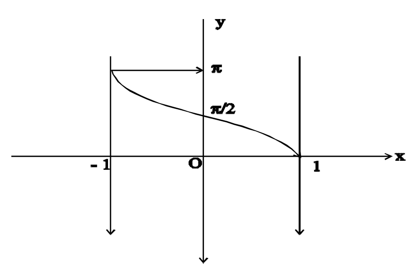
(vi) $\tan^{-1}x=\sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)=\cot^{-1}\left(\dfrac{1}{x}\right)=\sec^{-1}\left(\sqrt{1+x^2}\right)=\operatorname{cosec}^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

GRAPHS

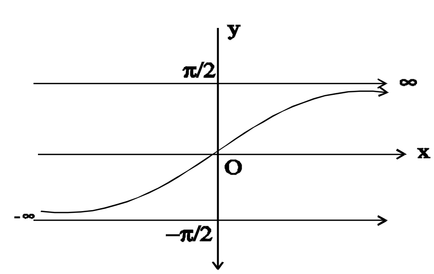
(1) $y=\sin^{-1}x, |x|\le 1,\ y\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$



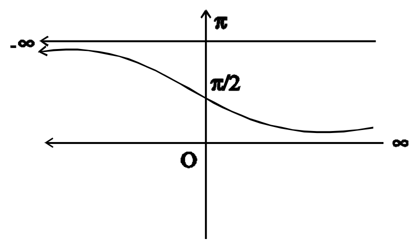
(2) $y=\cos^{-1}x, |x|\le 1,\ y\in[0,\pi]$



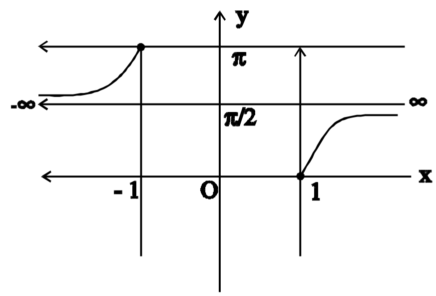
(3) $y=\tan^{-1}x,\ x\in R,\ y\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$



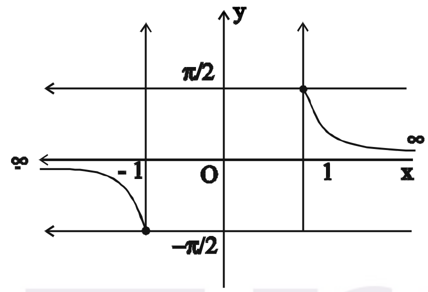
(4) $y = \cot^{-1} x,\ x \in R,\ y \in (0,\ \pi)$



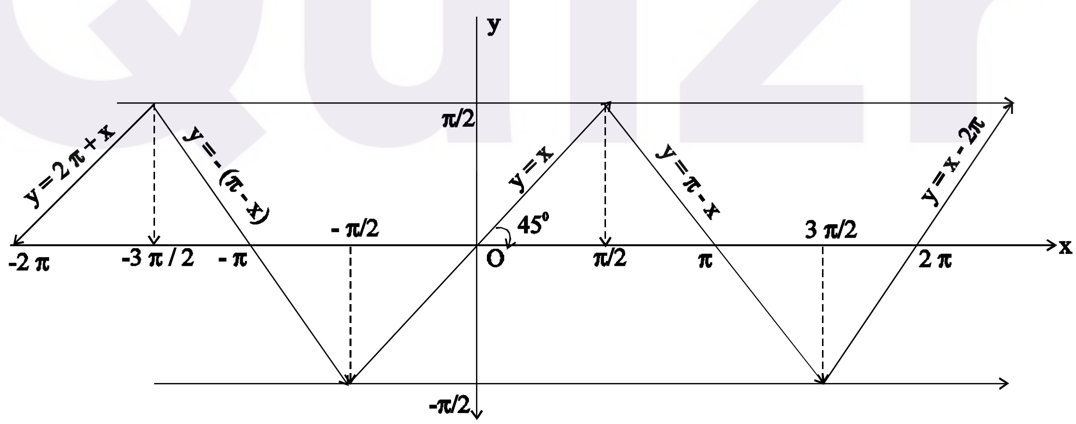
(5) $y = \sec^{-1} x. |x| \geq 1. y \in \left[0,\ \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\ \pi\right]$

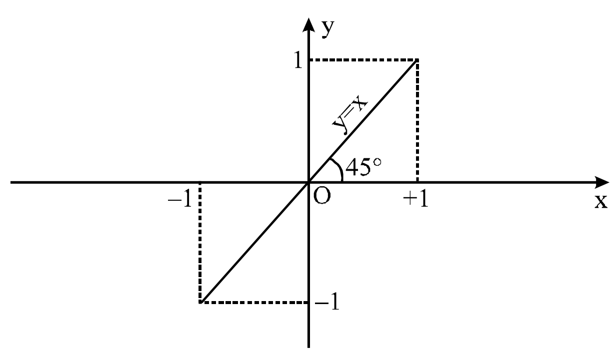


(6) $y = \operatorname{cosec}^{-1} x,\ |x| \geq 1, y \in \left[-\frac{\pi}{2},\ 0\right) \cup \left(0,\ \frac{\pi}{2}\right]$



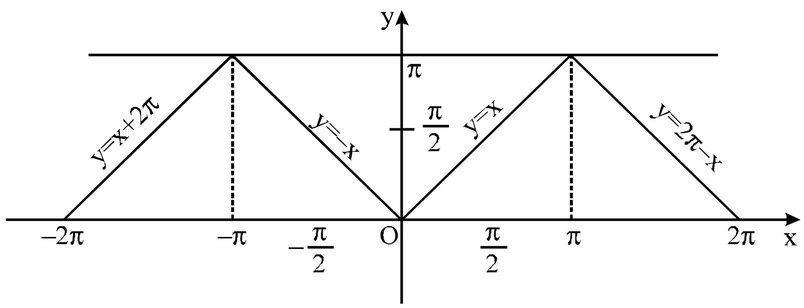
(i) $y = \sin^{-1}(\sin x) = x,\ x \in R,\ y \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$, y is periodic with period 2π



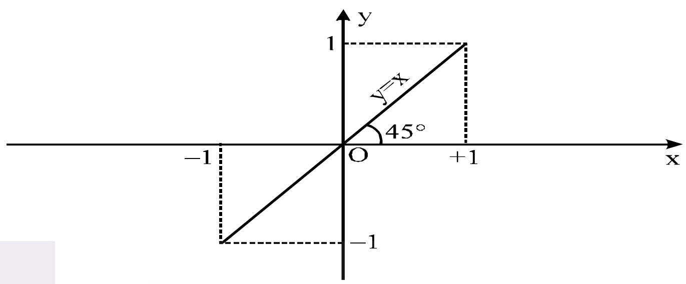


(ii) $y = \sin(\sin^{-1} x) = x,\ x \in [-1,\ 1],\ y \in [-1,\ 1]$

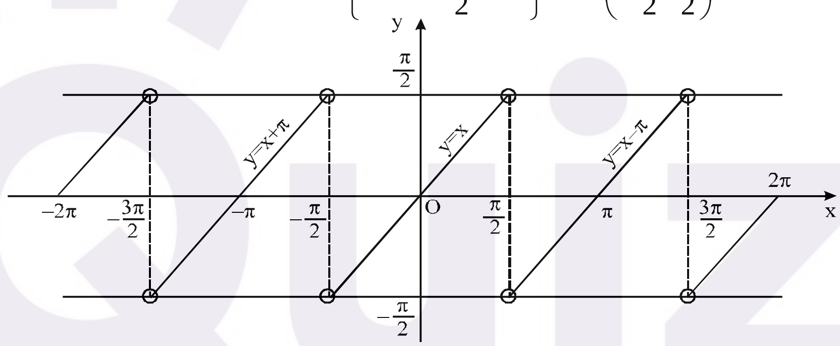
(i) $y = \cos^{-1}(\cos x) = x,\ x \in R,\ y \in [0,\ \pi]$, y is periodic with period 2π



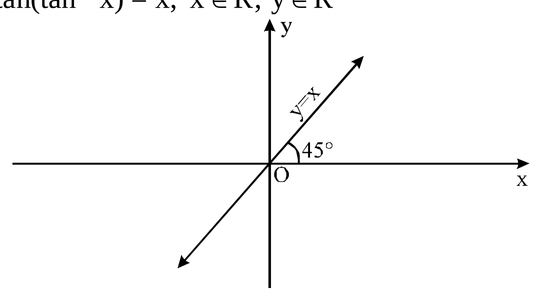
(ii) $y = \cos(\cos^{-1} x) = x,\ x \in [-1,\ 1],\ y \in [-1,\ 1]$



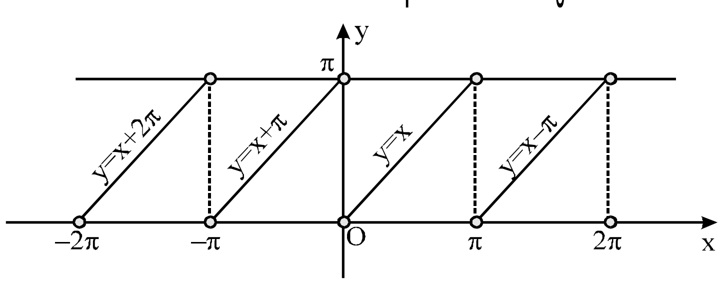
4.3 (i) $y = \tan^{-1}(\tan x) = x,\ x \in R - \left\{(2n-1)\frac{\pi}{2}\ n \in I\right\},\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, y is periodic with period π



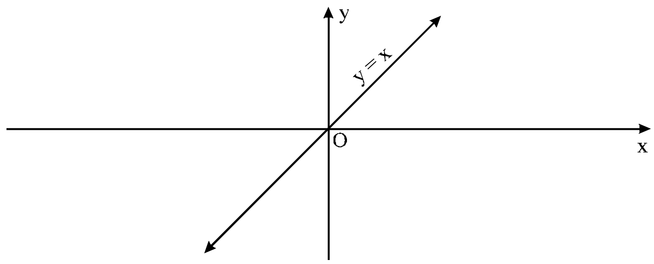
(ii) $y = \tan(\tan^{-1} x) = x,\ x \in R,\ y \in R$



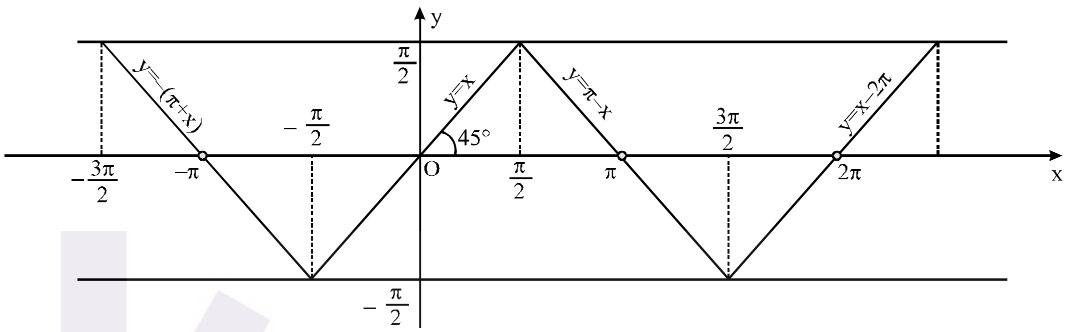
(i) $y = \cot^{-1}(\cot x) = x,\ x \in R - \{n\pi\},\ y \in (0,\ \pi)$, periodic with π



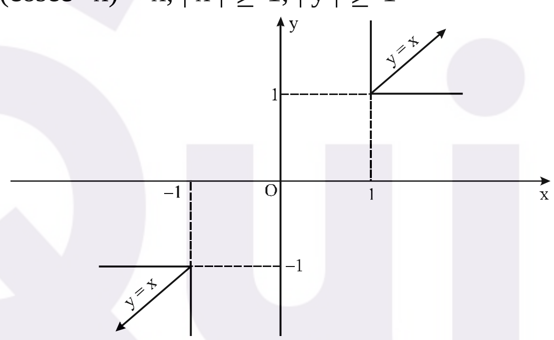
(ii) $y = \cot(\cot^{-1} x) = x\ ,\ x \in R,\ y \in R$



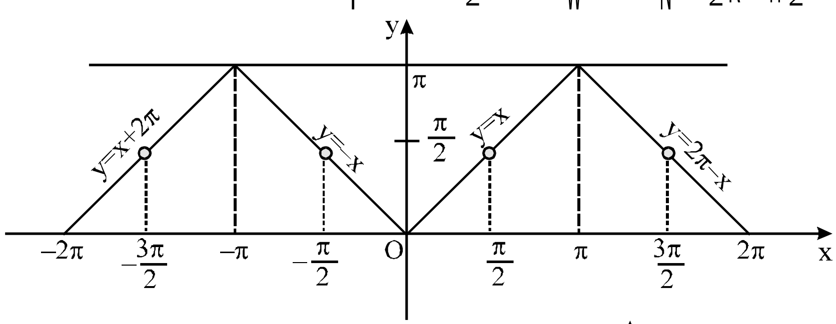
(i) $y = \mathrm{cosec}^{-1}(\mathrm{cosec}\ x) = x,\ x \in R - \{n\pi,\ n \in I\},\ y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, y is periodic with period 2π



(ii) $y = \mathrm{cosec}(\mathrm{cosec}^{-1} x) = x,\ |x| \geq 1,\ |y| \geq 1$



(i) $y = \sec^{-1}(\sec x) = x,\ x \in R - \left\{(2n-1)\frac{\pi}{2}\ \ n \in I\right\},\ \ y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, y is periodic with period 2π



(ii) $y = \sec(\sec^{-1} x) = x,\ |x| \geq 1;\ |y| \geq 1$